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SOLAR AND LUNAR TIDES IN MAGMA

B. V. Voitsekhovskii* and R. M. Garipov

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A planet consisting of an ideal incompressible fluid under the action of natural gravitation and attraction of celestial bodies is considered. The inhomogeneity and the presence of a solid core are modeled by a point mass concentrated at the center of the homogeneous planet. A formula is obtained that expresses the mass of the core in terms of the coefficient of flattening of the planet and the angular velocity of its rotation about its axis. The height of tides caused by the action of the Sun and satellites is determined. For the Earth, the numerical values have the same orders of magnitude as the height of tidal waves in the ocean. The initial assumption is valid if the energy of elastic deformation due to tidal forces is much smaller than the kinetic energy of the relative tidal motion of the planetary mass. This is shown to be typical of giant planets and, to a lesser degree, of the Earth. Exact Dirichlet, Riemann, and Ovsyannikov solutions and matrix calculus are used.

In the present work, we use the following scales: the units of length, density, and acceleration are, respectively, the mean radius r_0, the mean density ρ_0, and the free-fall acceleration on the planetary surface $(4\pi/3)\gamma\rho_0 r_0$, where γ is the gravitational constant. Hence we obtain the units of time $T = \sqrt{3/(4\pi\gamma\rho_0)}$ and velocities and r_0/T (for the Earth, 805 sec and 7.93 km/sec, respectively). In these units, the gravitational constant is equal to $3/(4\pi)$, and the planetary mass is $4\pi/3$. Therefore, the unit of mass is $r_0^3\rho_0$, and the unit of pressure is $\rho_0 r_0^2 T^{-2}$. Below, we use dimensionless variables, i.e., the ratios of the dimensional variables to their units of measure, retaining for them the same notation.

1. Natural Gravitation of the Planet. We first assume that the planet is homogeneous. Then, its density is equal to 1. It is assumed that the planet has the shape of an ellipsoid with center at the coordinate origin:

$$f(\boldsymbol{x}) \equiv \boldsymbol{x} \cdot A\,\boldsymbol{x} - 1 \leqslant 0,$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ is the radius-vector of a point in space and A is a symmetric, positive-definite matrix. Let the coordinate axes coincide with the axes of the ellipsoid. Then,

$$A = \begin{pmatrix} a_1^{-2} & 0 & 0 \\ 0 & a_2^{-2} & 0 \\ 0 & 0 & a_3^{-2} \end{pmatrix},$$

where a_1, a_2, and a_3 are the semiaxes of the ellipsoid. Since the mean radius of the ellipsoid is $\sqrt[3]{a_1 a_2 a_3} = 1$, it follows that $\det A = 1$. In particular, when the ellipsoid is a sphere, the matrix A is equal to unit matrix I.

*Deceased.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 41, No. 6, pp. 3–12, November–December, 2000. Original article submitted October 27, 1999; revision submitted January 10, 2000.

The potential of the natural gravitational field of the planet is denoted by $h_0(\boldsymbol{x})$. At point $\boldsymbol{x}$, a unit mass is acted upon by gravitational force $\nabla h_0(\boldsymbol{x})$ (∇ is the gradient). It should be noted that inside the ellipsoid, the natural gravitational potential is a polynomial of x_1, x_2, and x_3, which, in the present, specially oriented coordinate system, has the form

$$h_0(\boldsymbol{x}) = -(1/2)(d_1x_1^2 + d_2x_2^2 + d_3x_3^2 - d_0) \quad \text{for} \quad f(\boldsymbol{x}) \leqslant 0, \tag{1}$$

where

$$d_0 = \frac{3}{2}\int_0^\infty \Delta\, ds; \quad \Delta = \left(\prod_{i=1}^3 (1 + sa_i^{-2})\right)^{-1/2}; \quad d_i = \frac{3}{2}\int_0^\infty \frac{\Delta}{a_i^2 + s}\, ds \quad (i = 1,\, 2,\, 3)$$

(see [1, 2]). Outside the ellipsoid, the potential $h_0(\boldsymbol{x})$ is not expressed in terms of elementary functions.

The matrix

$$D_0 = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

is a function of the matrix A. It is easy to find the trace of this matrix — the sum of the diagonal elements $\operatorname{sp} D_0(A) = 3$ (integral of $-3d\Delta/ds$). This function is calculated approximately for the case where the ellipsoid is close to a sphere, i.e., $A \simeq I$:

$$D_0(A) = I + (3/5)(A - I) - (3/70)\operatorname{sp}(A - I)^2 I - (6/35)(A - I)^2 + O((A - I)^3).$$

The matrix function $D_0(A)$ does not depend on the choice of the coordinate system, and, since the last formula is invariant, it is also valid for the nondiagonal matrices A and D_0 if $\det A = 1$. Here terms of the second order of smallness are computed to see that they can be ignored. A linear approximation suffices to solve the formulated problem:

$$D_0(A) = (2/5)I + (3/5)A + O((A - I)^2). \tag{2}$$

In this approximation, $\det A \simeq 1 + \operatorname{sp}(A - I)$, whence, by virtue of the condition $\det A = 1$ it follows that $\operatorname{sp} A \simeq 3$.

Let the planet now have a solid core which contains fraction α of its mass. Then, the gravitational potential of the planet is equal to the sum of the potentials of the core, which can be approximately treated as point mass $(4\pi/3)\alpha$ located at the center, and a homogeneous fluid ellipsoid with density $1 - \alpha$:

$$h_0(\boldsymbol{x}) = \alpha|\boldsymbol{x}|^{-1} - (1/2)(1 - \alpha)(\boldsymbol{x} \cdot D_0\boldsymbol{x} - d_0),$$

where $|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2}$ is the length of the vector $\boldsymbol{x}$. The last formula is derived from formula (1). The gravitational potential of the point core is no longer a polynomial of the coordinates $\boldsymbol{x}$ inside the ellipsoid. However, with the adopted accuracy, it can be replaced by a polynomial in the neighborhood of the surface of the ellipsoid. Indeed, the equation of the surface $f(\boldsymbol{x}) = \boldsymbol{x} \cdot (A - I)\boldsymbol{x} + |\boldsymbol{x}|^2 - 1 = 0$ implies that on the surface, $|\boldsymbol{x}| = 1 + O(A - I)$. Therefore, with accuracy up to terms of the second order of smallness, $|\boldsymbol{x}|^{-1} = (1 + \boldsymbol{x} \cdot \boldsymbol{x} - 1)^{-1/2} \simeq 3/2 - (1/2)\boldsymbol{x} \cdot \boldsymbol{x}$ at $f(\boldsymbol{x}) = 0$.

Thus, with accuracy $O((A - I)^2)$, taking into account formula (2), we have

$$h_0(\boldsymbol{x}) \simeq -(1/2)\boldsymbol{x} \cdot D\,\boldsymbol{x} \qquad (|\boldsymbol{x}| \simeq 1), \tag{3}$$

where the terms that do not depend on $\boldsymbol{x}$ are dropped because only the gradient of h_0 is significant, and $D \simeq (1/5)(2 + 3\alpha)I + (3/5)(1 - \alpha)A$ ($\operatorname{sp} A \simeq 3$).

2. External Gravitation. We place the coordinate origin $\boldsymbol{x} = 0$ at the center of the planet, so that the Sun and planetary satellites rotate around it in specified paths by virtue of Kepler's laws. This implies that we are in the Ptolemaic system. The only difference is that our coordinate system does not rotate, and, therefore, stars appear fixed. The planet is acted upon by gravitational forces from celestial bodies with potential $h(\boldsymbol{x}, t)$ that depends explicitly on the time t. Since the radius of the planet is much smaller than the distances to these bodies, the function $h(\boldsymbol{x}, t)$ is approximated by the Taylor formula

$$h(\boldsymbol{x}, t) \simeq h(0, t) + \sum_{i=1}^{3} \frac{\partial h(0, t)}{\partial x_i} x_i + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 h(0, t)}{\partial x_i \, \partial x_j} x_i x_j.$$

The laws of dynamics on the planet, along with the gravitational forces, contain inertial forces whose potential depends linearly on $\boldsymbol{x}$ because of the translational character of motion of our coordinate system. The inertial-force potential and the first-order terms of the Taylor expansion of the function $h(\boldsymbol{x}, t)$ cancel out if we take into account that the planet moves in space as a mass point. The zero term $h(0, t)$ is insignificant. Thus, the planetary mass moves in the same manner as in an inertial coordinate system in the presence of an external-force field with potential

$$h_b(\boldsymbol{x}, t) = \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 h(0, t)}{\partial x_i \, \partial x_j} x_i x_j. \tag{4}$$

For the solution of our problem, it is of fundamental importance that this potential is a second-order polynomial in $\boldsymbol{x}$.

The gravitational fields of different bodies are summed up. Therefore, it suffices to calculate expression (4) for one celestial body. Let a celestial body with mass m move along the trajectory $\boldsymbol{x} = \boldsymbol{R}(t)$. The vector function of time $\boldsymbol{R}(t)$ is uniquely determined by Kepler's laws. According to the gravitation law, this body produces a gravitational field with potential $h(\boldsymbol{x}, t) = \gamma m |\boldsymbol{x} - \boldsymbol{R}(t)|^{-1}$. We find the derivatives

$$\frac{\partial^2 h(0, t)}{\partial x_i \, \partial x_j} = c(t)(3e_i(t)e_j(t) - \delta_{ij}) \qquad (i,\ j = 1,\ 2,\ 3),$$

where $c(t) = \gamma m |\boldsymbol{R}(t)|^{-3}$, $\boldsymbol{e}(t) = (e_1(t), e_2(t), e_3(t)) = \boldsymbol{R}(t)/|\boldsymbol{R}(t)|$, $\delta_{ii} = 1$, and $\delta_{ij} = 0$ $(i \neq j)$ is the Kronecker delta. The unit vector $\boldsymbol{e}(t)$ is directed toward the celestial body. This expression with two subscripts is treated as an element of the matrix $E(t)$. Then, the external-gravitation potential (4) is written as

$$h_b(\boldsymbol{x}, t) = (1/2)\boldsymbol{x} \cdot E(t)\boldsymbol{x}. \tag{5}$$

If there are several celestial bodies, their associated matrices $E(t)$ should be summed up. We note that $\operatorname{sp} E(t) = 0$.

If a celestial body moves in a circular path $|\boldsymbol{R}(t)| = \text{const}$ with constant angular velocity χ, then equating the mutual gravitational attraction force to the centrifugal force (in this case, the distance should be reckoned from the center of mass), we obtain $c(t) = (m/(m + m_0))\chi^2$, where m_0 is the planetary mass.

As an example, we consider the Earth, which is acted upon by the gravitational fields of two celestial bodies (the Sun and the moon), which rotate in almost circular paths with angular velocities $\chi_1 = 1,6 \cdot 10^{-4}$ and $\chi_2 = 2,14 \cdot 10^{-3}$, respectively. The angle between the planes of their paths is only 5°, and, hence, to simplify the above formulas, we assume that these celestial bodies move in the same plane. We direct the x_3 axis perpendicular to this plane toward the North Pole of the Earth. Then, the vectors determining the directions to the celestial bodies and the external-gravitation matrix have the form

$$\begin{gathered} \boldsymbol{e}^k(t) = (\cos(\chi_k t + \lambda_k), \sin(\chi_k t + \lambda_k), 0) \qquad (k = 1,\ 2), \\ E(t) = \sum_{k=1}^{2} c_k (3e_i^k(t)e_j^k(t) - \delta_{ij}) \qquad (c_1 = \chi_1^2, \quad c_2 = \chi_2^2/82). \end{gathered} \tag{6}$$

Here $k = 1$ corresponds to the Sun and $k = 2$ corresponds to the moon. The constants λ_1 and λ_2 are determined by the choice of the direction of the coordinate axis x_1 (on the plane of the paths) and the beginning of the time readout. They can be set equal to zero: $\lambda_1 = \lambda_2 = 0$.

3. Equations of Motion and Conservation Laws. The planetary material is considered an ideal and incompressible homogeneous fluid, whose motion is described by the combined Euler equations

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla(p - h) = 0,$$

where $\boldsymbol{u}$ is the fluid velocity, p is the pressure, and $h = h_0 + h_b$ is the potential of the gravitational forces (3) and (5). If the fluid density is not equal to unity (in the presence of the planet's core), then p denotes the pressure divided by the density and is also called the pressure. In this case, the reasoning given below remains in force. As was first noted by Dirichlet, a solution of the form $\boldsymbol{u} = B(t)\boldsymbol{x}$ [$B(t)$ is the required matrix function of time t] satisfies all the necessary boundary conditions. Below, we consider only this solution. From the second Euler equation it follows that the difference $p - h$ is a second-order polynomial in the coordinates of the vector $\boldsymbol{x}$ inside the ellipsoid, although in the presence of the planet's core, each of the functions p and h is not a polynomial.

On the planetary surface, the pressure is equal to atmospheric pressure: $p = p_0$, which can be considered identical everywhere, i.e., independent of $\boldsymbol{x}$. Therefore, $p - h = p_0 + (1/2)\boldsymbol{x} \cdot C\boldsymbol{x}$ for $f \equiv \boldsymbol{x} \cdot A\boldsymbol{x} - 1 = 0$, where $C = D(A) - E(t)$. From this it follows that zeros of the second-order polynomial f in $\boldsymbol{x}$ that does not have multiple roots are also zeros of the polynomial $p - h - p_0 - (1/2)\boldsymbol{x} \cdot C\boldsymbol{x}$ of the same order. Hence, there is a scalar λ such that $p - h - p_0 - (1/2)\boldsymbol{x} \cdot C\boldsymbol{x} = -(\lambda/2)f$ for $f \leqslant 0$. From this, we find $p - h$ inside the ellipsoid: $p - h = p_0 + \lambda/2 + (1/2)\boldsymbol{x} \cdot C\boldsymbol{x} - (\lambda/2)\boldsymbol{x} \cdot A\boldsymbol{x}$ for $f \leqslant 0$. Substituting $\boldsymbol{u} = B(t)\boldsymbol{x}$ and the obtained expression for $p - h$ into the second Euler equation, we obtain $\dot{B}\boldsymbol{x} + (B\boldsymbol{x} \cdot \nabla)B\boldsymbol{x} + C\boldsymbol{x} - \lambda A\boldsymbol{x} = 0$, where the dot from above denotes differentiation with respect to time. Here the second term is equal to $B^2\boldsymbol{x}$, which is easy to see by computing the ith coordinate. Since the obtained equality is valid for all radius-vectors $\boldsymbol{x}$ of points of the ellipsoid, it is equivalent to the matrix equality

$$\dot{B} + B^2 + C - \lambda A = 0. \tag{7}$$

Next, from the first Euler equation (which implies the incompressibility of the fluid) it follows that $\operatorname{sp} B = 0$. The unknown quantity λ is obtained from the condition

$$\lambda = \operatorname{sp}(B^2 + C)/\operatorname{sp} A = (\operatorname{sp} B^2 + 3)/\operatorname{sp} A. \tag{8}$$

It remains to satisfy the kinematic condition of nonpenetration through the fluid surface:

$$\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f = 0 \qquad \text{for} \quad f = 0.$$

For the same reason as above, the polynomial on the left side of this equality is a multiple of f, i.e., there is a scalar μ such that for all $\boldsymbol{x}$, the equality $\boldsymbol{x} \cdot \dot{A}\boldsymbol{x} + B\boldsymbol{x} \cdot 2A\boldsymbol{x} = \mu(\boldsymbol{x} \cdot A\boldsymbol{x} - 1)$ is valid. Equating the zero-order monomials, we obtain $\mu = 0$. We symmetrize the matrix of the second quadratic form: $B\boldsymbol{x} \cdot 2A\boldsymbol{x} = \boldsymbol{x} \cdot 2B^*A\boldsymbol{x} = \boldsymbol{x} \cdot (B^*A + AB)\boldsymbol{x}$, where the asterisk denotes a conjugate matrix. We recall that the matrix A is symmetric: $A^* = A$. The matrix of the obtained zeroth quadratic form should be equal to zero:

$$\dot{A} + AB + B^*A = 0. \tag{9}$$

We have obtained the closed system (7)–(9) for matrices A and B, since the matrix C is a given function of A and t. This system of equations is considered on the manifold $\det A = 1$, $\operatorname{sp} B = 0$, and, hence, it has order 13. In the derivation of these equations, it was assumed that $A \simeq I$. However, in the absence of the planet's core, this *a priori* constraint can be dropped. Then, it is necessary to use the exact expression (1) for the function $D(A) = D_0(A)$.

Let us now write the conservation laws for the homogeneous planet ($\alpha = 0$) in matrix form. We determine the circulation $\Gamma = \operatorname{sp}((B^* - B)A^{-1})^2$, the angular momentum $L = BA^{-1} - A^{-1}B^*$, the moment of external forces $M = EA^{-1} - A^{-1}E$, and the energy $H = (1/2)\operatorname{sp}(B^*BA^{-1} - EA^{-1}) - d_0$, where d_0 is a function of the semiaxes of the ellipsoid from formula (1). These quantities coincide with the conventional quantities with accuracy up to numerical coefficients. For example, the antisymmetric matrix L determines the vector (L_{32}, L_{13}, L_{21}), which is equal to the total angular momentum of the fluid divided by $4\pi/3$. By virtue of Eqs. (7)–(9), the known conservation laws become

$$\Gamma = \text{const}, \qquad \dot{L} = M, \qquad \dot{H} = (1/2)\,\mathrm{sp}\,(-\dot{E}A^{-1}). \tag{10}$$

4. Flattening of the Planet. We use Eqs. (7)–(9) to evaluate the coefficient of flattening of the planet because of its rotation about its axis. In this case, it is possible to ignore external gravitation: $E = 0$. We find a solution that describes rigid rotation of an axisymmetric ellipsoid with angular velocity ω about of its symmetry axis. Let the symmetry axis be the coordinate axis x_3:

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \quad (A_1^2 A_3 = 1), \qquad B = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, because the matrix A is permutable with B, Eq. (9) is an identity, and (7) becomes

$$B^2 + D(A) - \lambda A = 0 \tag{11}$$

[relation (8) is a consequence of this equation].

The system of transcendental equations (11) for the quantities A_1 and λ ($A_3 = A_1^{-2}$) with specified velocity of rotation ω was solved by Maclaurin in the absence of the planet's core [$D(A) = D_0(A)$] but without assuming the smallness of $A - I$. Then, the function $D_0(A)$ is defined by formula (1). Maclaurin established that at $\omega < 0.58\ldots$ there are two solutions. At $\omega \to 0$, one of these becomes a sphere, and the other is flattened out to become an infinitely thin disk. For $\omega > 0.58\ldots$, real solutions do not exist.

We find a solution close to a sphere using the approximate expression (3) for the function $D(A)$. For the Earth, $\omega = 0.0588$, i.e., is rather small. Therefore, ω^2 is treated as a small parameter. At $\omega^2 = 0$, system (11) has a solution $A = I$, $\lambda = 1$. Subtracting the third term from the first diagonal element of the matrix equation, we obtain $-\omega^2 + ((3/5)(1-\alpha) - \lambda)(A_1 - A_3) = 0$. From this equation it follows that $A_1 - A_3$ is a quantity of the first order of smallness, so that it suffices to calculate the factor of this quantity with accuracy up to zero-order terms, i.e., to set it equal to $(3/5)(1-\alpha) - 1$. Then, $A_3 - A_1 = (5/(2+3\alpha))\omega^2 + O(\omega^4)$. By definition, the flattening coefficient for the planet is equal to $\delta = 1 - (A_1/A_3)^{1/2} = (A_3 - A_1)/(A_3 + \sqrt{A_1 A_3})$. Here the coefficient at $A_3 - A_1$ can be evaluated with accuracy up to zero-order terms. Finally, we obtain

$$\delta = (5/(4+6\alpha))\omega^2. \tag{12}$$

The masses of the planetary cores can be evaluated from formula (12). For example, for the Earth, knowing $\delta = 1/298$ and ω, we obtain $\alpha = 0.19$. Hence, about one-fifth of the Earth's mass is concentrated in the core.

5. Height of Tides. Linearizing system (7)–(9) in the neighborhood of rest $A = I$ and $B = 0$, we obtain

$$\dot{B} - ((2+3\alpha)/5)A' = E(t), \qquad \dot{A}' + B + B^* = 0,$$

where $A' = A - I$, $\mathrm{sp}\,A' = 0$, and $\mathrm{sp}\,B = 0$. We expand the matrix B in the sum of the symmetric F and antisymmetric Ω parts: $B = F + \Omega$, $F = (1/2)(B + B^*)$, and $\Omega = (1/2)(B - B^*)$. The matrix Ω specifies the velocity field for a fluid rotating as a rigid body, and F is the strain-rate field. Accordingly, splitting the first equation into two equations — antisymmetric and symmetric — we obtain

$$\dot{\Omega} = 0, \qquad \dot{F} - ((2+3\alpha)/5)A' = E(t), \qquad \dot{A}' + 2F = 0. \tag{13}$$

Hence it follows that the velocity of rigid rotation Ω is constant and arbitrary and does not influence the deformation of the planet described by the matrices A' and F.

We consider the last two equations of system (13). Eliminating F, we obtain $\ddot{A}' + ((4+6\alpha)/5)A' = -2E(t)$. From this, we find the frequency of natural oscillations of the planet $\sigma = \sqrt{(4+6\alpha)/5}$. This is a very high frequency. For the Earth, $\sigma = 1.01$ and the dimensional period of these oscillations is 1.39 h. Since the frequencies of the function $E(t)$ are much lower than $\sigma = 1.01$, ignoring the term $\ddot{A}'$ in the last equation for A', we obtain the following expression for forced oscillations:

$$A' = -(5/(2 + 3\alpha))E(t). \tag{14}$$

Remark 1. System (13) is considered on the manifold $\operatorname{sp} A' = 0$, $\operatorname{sp} F = 0$, on which it has order 13. System (13) has the following eigenfrequencies: 0 of multiplicity 3 and $\pm\sigma$, each of multiplicity 5. Suppose we have performed linearization in the neighborhood of rigid rotation with small angular velocity ω. Then, the eigenfrequencies will be functions of the parameter ω. At $\omega = 0$, we obtain the same frequencies as above. For $\omega \neq 0$, they, generally speaking, split and their number becomes equal to their multiplicity. Because of the law of conservation of angular momentum (10) (at $E = 0$), the frequency 0 splits into three components: 0 and $\pm\omega$. The ten components resulting from splitting of the frequencies $\pm\sigma$ can play a role in nonlinear effects.

Let $\boldsymbol{x} = (1 + \eta)\boldsymbol{a}$ be a point on the planetary surface, where $\boldsymbol{a} = (\cos\lambda\cos\beta, \sin\lambda\cos\beta, \sin\beta)$ is a unit vector, so that η denotes the vertical departure of the planetary surface from its equilibrium position under the action of celestial bodies. To express η in terms of the matrix A', we substitute the expression for $\boldsymbol{x}$ into the equation of the planetary surface $\boldsymbol{x} \cdot (I + A')\boldsymbol{x} - 1 = 0$ and ignore terms of order η^2. With allowance for formula (14) we obtain

$$\eta = -(1/2)\boldsymbol{a} \cdot A'\boldsymbol{a} = (5/(4 + 6\alpha))\boldsymbol{a} \cdot E(t)\boldsymbol{a}. \tag{15}$$

Let us consider the Earth. In astronomy, λ and β are termed ecliptic coordinates. In contrast to the geographical coordinate system, the Earth's surface rotates about the ecliptic system with a period equal to 24 h.

Substituting expression (6) for the matrix $E(t)$ into formula (15), we obtain

$$\eta = \frac{5}{4 + 6\alpha}\sum_{k=1}^{2} c_k(3(\boldsymbol{e}^k \cdot \boldsymbol{a})^2 - 1) = \frac{5}{4 + 6\alpha}\sum_{k=1}^{2} c_k(3\cos^2\beta\cos^2(\chi_k t + \lambda_k - \lambda) - 1). \tag{16}$$

This expression, as a function of the ecliptic longitude λ, has period π, i.e., on the ecliptic parallel there are two maxima and two minima. Because of the Earth's rotation, these humps move over its surface, forming tides two times every 24 h. In addition, they drift slowly about the ecliptic coordinate system because of the motion of celestial bodies along their trajectories. To determine the height of these humps, we bring expression (16) to the form

$$b_1\cos(2\lambda) + b_2\sin(2\lambda) + b_3 = \sqrt{b_1^2 + b_2^2}\cos(2\lambda - \varphi) + b_3,$$

where $\varphi = \arctan(b_2/b_1)$. Hence it follows that the height of a tide (amplitude of the tidal wave) is equal to

$$b = \sqrt{b_1^2 + b_2^2} = (15/(8 + 12\alpha))\cos^2\beta(c_1^2 + c_2^2 + 2c_1c_2\cos(2(\chi_1 - \chi_2)t + 2\lambda_1 - 2\lambda_2))^{1/2}.$$

Depending on the day of the month, the height of a tide varies from $q|c_1 - c_2|$ to $q(c_1 + c_2)$, where $q = (15/(8 + 12\alpha))\cos^2\beta$. At $\beta = 0$, i.e., on the ecliptic equator, these quantities are equal to 0.28 and 0.76 m, respectively. To determine the height of a tide at a particular point on the Earth's surface at a given time, it is necessary to express the ecliptic coordinates in terms of the geographical ones and use the general formula (16). In the Earth's crust, tides have an average amplitude of 0.5 m, which is in good agreement with the theoretical result obtained.

6. Estimate of the Elastic Strain Energy. We now compare the kinetic energy K of tidal motion with the energy U of elastic strain of the planet under the action of celestial bodies. If $K \gg U$, the planet can be considered fluid. Since the Reynolds number is large and all condensed media are insignificantly compressible, it is reasonable to consider this fluid ideal and incompressible. This justifies the applicability of the theoretical results for the description of astronomical phenomena. Since the results given in the present section have the character of estimates, we shall not take into account the inhomogeneity of the planet (set $\alpha = 0$).

In the evaluation of K and U, it is necessary to consider tidal motion relative to the geographical coordinate system $\boldsymbol{x}'$ that rotates together with the planet around its axis in order to eliminate rigid rotation and take into account only strains. The kinetic energy K of the relative motion depends on the velocity of rotation of the planet about its axis. Indeed, a point on the surface of the planet moves vertically up and down at the height of the tidal hump b in time Δt, during which the hump passes through it. Since the two humps are fixed relative to celestial bodies, which move slowly over the sky, then Δt is equal to the time of half-rotation of the planet: $\Delta t = \pi/\omega$, where ω is the angular velocity of planetary rotation. Thus, the relative rate of tidal motion has order of magnitude $b/\Delta t = b\omega/\pi$. From this, $K \sim b^2\omega^2$. At the same time, the strain of the planet has order $b/1$ (mean radius of the planet is set equal to 1). Therefore, $U \sim Gb^2$, where G is the shear modulus of the planetary material (because of the incompressibility, the strain energy does not depend on the second elastic constant). Thus,

$$U/K = æG/\omega^2, \tag{17}$$

where æ is a numerical coefficient. Here K and U are some characteristic values and not functions of time. In this formula, we convert to dimensional quantities: $U/K = æG/(\rho_0 r_0^2\omega^2)$. For the Earth, the quantity $G_0 = \rho_0 r_0^2\omega^2$ is equal to 1.2 GPa. As G it is necessary to use the shear modulus of magma, which is unknown. For comparison, we give the shear modulus of aluminum, $G = 24.5$ GPa. The shear modulus of magma is apparently much smaller. For Jupiter, $G_0 = 204$ GPa. In fact, the last formula gives an answer to the question posed, because, usually, $æ \sim 1$ (we shall see this below when calculating the coefficient æ).

It should be noted that the value of the parameter G_0 does not depend on the value of tidal forces. The ratio G/G_0 characterizes the role of elastic and inertial forces in any motion of a planetary scale, where the wave length is comparable to the radius of the planet. In this case, in the expression for G_0, ω must be replaced by the characteristic frequency of motion. In particular, for the above natural oscillations of the planet with a high frequency σ, the value of G_0 is much larger than G (for the Earth, 354 GPa). Therefore, the role of elastic forces is even less significant. Short seismic waves are apparently beyond the scope of the model considered in the present paper.

We attach the coordinate system (x_1', x_2', x_3') to the planet and bring the x_3' axis in coincidence with the rotation axis of the planet. The angle between the x_3 and x_3' axes (orbit and equatorial planes) is denoted by θ. For the Earth, $\theta = 23.5°$. The relationship between the former and new coordinates of the same point in space is specified by the orthogonal matrix: $\boldsymbol{x} = Z(t)\boldsymbol{x}'$. We can write $Z(t) = Z_0 Z_1(t)$, where Z_0 is a constant orthogonal matrix and $Z_1(t)$ is the rotation about the x_3' axis at angle ωt. Therefore,

$$Z^{-1}\dot{Z} = Z_1^{-1}\dot{Z}_1 = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \Omega'.$$

By virtue of the permutability of the matrices $Z_1(t)$ and $\dot{Z}_1(t)$, we have $\Omega = Z(t)\Omega' Z(t)^{-1} = \dot{Z}(t)Z(t)^{-1} = Z_0\Omega' Z_0^{-1}$. The antisymmetric matrices Ω and Ω' specify the angular-velocity vector of planetary rotation in the coordinate systems $\boldsymbol{x}$ and $\boldsymbol{x}'$, respectively.

Let us determine how the matrices A and B are converted by the replacement of coordinates $\boldsymbol{x} \to \boldsymbol{x}'$. Substituting $\boldsymbol{x} = Z\boldsymbol{x}'$ into the equation of the planetary surface, we obtain

$$\boldsymbol{x} \cdot A\boldsymbol{x} - 1 = Z\boldsymbol{x}' \cdot AZ\boldsymbol{x}' - 1 = \boldsymbol{x}' \cdot Z^* AZ\boldsymbol{x}' - 1 = 0.$$

Hence it follows that $A \to A' = Z^*AZ = Z^{-1}AZ$ (for the orthogonal matrices $Z^* = Z^{-1}$). To find the form of the matrix B' in the coordinate system $\boldsymbol{x}'$, we recall that $B\boldsymbol{x}$ is the fluid velocity:

$$\boldsymbol{u} = \dot{\boldsymbol{x}} = \frac{d(Z\boldsymbol{x}')}{dt} = Z\dot{\boldsymbol{x}}' + \dot{Z}\boldsymbol{x}' = B\boldsymbol{x}.$$

From this, we obtain the relative velocity of the fluid $\boldsymbol{u}' = \dot{\boldsymbol{x}}' = Z^{-1}(B\boldsymbol{x} - \dot{Z}\boldsymbol{x}') = (Z^{-1}BZ - Z^{-1}\dot{Z})\,\boldsymbol{x}'$. It is obvious that it is also a linear function of the coordinates: $\boldsymbol{u}' = B'\boldsymbol{x}'$, where $B' = Z^{-1}BZ - Z^{-1}\dot{Z} = Z^{-1}BZ - \Omega'$.

Multiplying the left side of Eqs. (7) and (9) by Z^{-1} and the right side by Z and substituting $A = ZA'Z^{-1}$ and $B = Z(B' + \Omega')Z^{-1}$, we readily obtain

$$\dot{B}' + B'^2 + 2\Omega' B' + \Omega'^2 + D_0(A') - E'(t) - \lambda A' = 0, \quad \dot{A}' + A'B' + B'^* A' = 0,$$

where $E' = Z^{-1}EZ$ and $\lambda = (\mathrm{sp}\,(B'+\Omega')^2+3)/\mathrm{sp}\,A'$. Linearizing this system of equations in the neighborhood of the solution $A' = I$, $B' = 0$, and assuming that the quantity Ω' is small, we have a system of equations that coincides with (13) for $\alpha = 0$. Therefore, the antisymmetric part of the matrix B' specifies constant rotation, which now should be absent: $B'^* = B'$. Although the frequency of the function $E'(t)$, which is approximately equal to 2ω, is much higher than that of $E(t)$, it is nevertheless much smaller than the eigenfrequency $2/\sqrt{5}$. Thus, we obtain

$$B' = (5/4)\dot{E}'(t). \tag{18}$$

Let us calculate the kinetic energy of the relative motion:

$$K = \frac{1}{2} \int\limits_{f \leqslant 0} |\boldsymbol{u}'|^2 \, d\boldsymbol{x}' = \frac{2\pi}{3} \, \mathrm{sp}\,(B'^* B' A'^{-1}).$$

In the linear approximation considered, we simplify this expression with allowance for (18):

$$K = \frac{2\pi}{3} \, \mathrm{sp}\,(B'^* B') = \frac{2\pi}{3} \, \mathrm{sp}\, \Big(\frac{5}{4} \, \dot{E}'(t)\Big)^2. \tag{19}$$

We note that the trace of the square of a symmetric matrix is equal to the sum of the squares of its elements.

We now find the elastic strain energy. The trajectories of points of the planet are calculated from the fluid-velocity field obtained. They satisfy the differential equation

$$\dot{\boldsymbol{x}}' = \boldsymbol{u}' = B'(t)\boldsymbol{x}', \qquad \boldsymbol{x}'\Big|_{t=t_0} = \boldsymbol{\xi},$$

where t_0 is a certain time. Since the displacements $\boldsymbol{x}' - \boldsymbol{\xi}$ are small, the method of successive approximations is applicable. As a first approximation, taking into account (18), we have

$$\dot{\boldsymbol{x}}' = B'(t)\boldsymbol{\xi} = (5/4)\dot{E}'(t)\boldsymbol{\xi}, \qquad \boldsymbol{x}' - \boldsymbol{\xi} = (5/4)(E'(t) - E'(t_0))\boldsymbol{\xi}.$$

It is reasonable to reckon the displacement of a point from its hypothetical position $\boldsymbol{\xi}$ where there was no deformation of the planet, i.e., $E'(t_0) = 0$. From this, we obtain the strain tensor, which is the symmetric matrix

$$\varepsilon = \frac{1}{2}\Big(\frac{\partial(x_i' - \xi_i)}{\partial \xi_j} + \frac{\partial(x_j' - \xi_j)}{\partial \xi_i}\Big) = \frac{5}{4} \, E'(t).$$

The elastic-strain energy is given by

$$U = \int\limits_{f \leqslant 0} \Big(\frac{\lambda}{2} \, (\mathrm{sp}\,\varepsilon)^2 + \frac{\mu}{2} \, \mathrm{sp}\,\varepsilon^2\Big) d\boldsymbol{x}',$$

where λ and μ are the Lamé coefficients. Since $E'(t)$ does not have traces, the first term in the integrand vanishes ($\mathrm{sp}\,\varepsilon = 0$). Since the strain tensor ε does not depend on $\boldsymbol{x}'$, we find that

$$U = \frac{2\pi\mu}{3} \, \mathrm{sp}\, \Big(\frac{5}{4} \, E'(t)\Big)^2. \tag{20}$$

Thus, by virtue of (19) and (20), we have

$$\frac{U}{K} = \mu \, \frac{\mathrm{sp}\, E'(t)^2}{\mathrm{sp}\, \dot{E}'(t)^2}. \tag{21}$$

Let us consider the case where just one celestial body rotates around the planet in a circular orbit with angular velocity χ. Then, the denominator of the ratio (21) is evaluated directly [see (6)]: $\mathrm{sp}\, E'(t)^2 = \mathrm{sp}\, E(t)^2 = 6c^2$. We convert the denominator, taking into account the inequality $\chi \ll \omega$:

TABLE 1

Planet	α	G_0, GPa
Venus	—	$2 \cdot 10^{-4}$
Earth	0.19	1.19
Mars	0.06	0.23
Jupiter	0.47	204
Saturn	0.65	68
Uranus	<0	6
Neptune	0.57	21

$$\dot{E}' \simeq \dot{\overline{Z^{-1}}} EZ + Z^{-1}E\dot{Z} = Z^{-1}(Z\dot{\overline{Z^{-1}}}E + E\dot{Z}Z^{-1})Z = Z^{-1}(-\Omega E + E\Omega)Z.$$

Here we have used the equality $Z\dot{\overline{Z^{-1}}} = -\dot{Z}Z^{-1}$, which is obtained by differentiation of the identity $ZZ^{-1} = I$ by t. Thus,

$$\operatorname{sp}\dot{E}'(t)^2 = \operatorname{sp}(\Omega E - E\Omega)^2 = 9c^2 \operatorname{sp}((\Omega\boldsymbol{e})_i e_j + e_i(\Omega\boldsymbol{e})_j)^2 = 18c^2|\Omega\boldsymbol{e}|^2 \leqslant 18c^2\omega^2.$$

For $\theta = 0$ the above formula is an equality, and, hence, $U/K = \mu/(3\omega^2)$. Because $\mu = 2G$, this leads to formula (17) with a constant $æ = 2/3$.

Thus, the ratio G/G_0 shows that elastic forces play a less important role in tidal motions of planets than inertial forces. The values of the parameter G_0 and the values computed from formula (12) for the fraction α of masses of the planetary cores are listed in Table 1. The data are taken from [3]. For Uranus, the value $\alpha < 0$ is obtained, which has no physical meaning. This means that measurements of the angular velocity of rotation and the flattening coefficient are inaccurate or the flattening of this planet does not result from its rotation about its axis.

REFERENCES

1. B. Riemann, "Beitrag zu den Untersuchungen über die Bewegung eines flüssigen gleichartige Ellipsoides," *Gött. Abh.*, **3**, No. 3 (1860).
2. L. Ovsyannikov, *Problem of Unsteady Motion of a Fluid with a Free Boundary: General Equations and Examples* [in Russian], Nauka, Novosibirsk (1967).
3. I. K. Kikoin (ed.), *Tables of Physical Quantities, Handbook*, Atomizdat, Moscow (1976).